A Topological Representation Theorem for Tropical Oriented Matroids: Part I

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Tropical oriented matroids were defined by Ardila and Develin in 2007 in analogy to (classical) oriented matroids. In this paper we present one tropical analogue for the Topological Representation Theorem.

1 Introduction

Oriented matroids abstract the combinatorial properties of arrangements of real hyperplanes and are ubiquitous in combinatorics. In fact, an arrangement of n (oriented) real hyperplanes in $\mathbb{R}^d$ induces a regular cell decomposition of $\mathbb{R}^d$. Then the covectors of the associated oriented matroid encode the position of the points of $\mathbb{R}^d$ (respectively, the cells in the subdivision) relative to the each of the hyperplanes in the arrangement. It turns out though that there are oriented matroids which cannot be realised by any arrangement of hyperplanes. The famous Topological Representation Theorem by Folkman and Lawrence [FL78] (see also [BLS+99]), however, states that every oriented matroid can be realised as an arrangement of PL-*pseudo*hyperplanes.

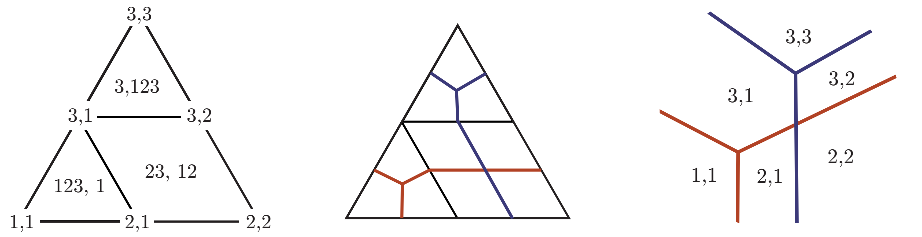
In this paper, we will study a *tropical* analogue of oriented matroids.

Tropical geometry is the algebraic geometry over the tropical semiring ($\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}, \oplus, \otimes$), where

$$\oplus : \bar{\mathbb{R}} \times \bar{\mathbb{R}} \to \bar{\mathbb{R}} : a \oplus b := \min\{a, b\} \qquad \text{and} \qquad \otimes : \bar{\mathbb{R}} \times \bar{\mathbb{R}} \to \bar{\mathbb{R}} : a \otimes b := a + b$$

are the tropical addition and multiplication. A tropical hyperplane is the vanishing locus of a linear tropical polynomial, *i.e.,* the set of points x where the minimum $p(x) = \bigoplus(a_i \otimes x_i)$ is attained at least twice.

Tropical geometry can be seen as a piecewise linear image of classical algebraic geometry and has in the past years received attention since there are often strong relationships



(a) A (regular) mixed subdivision of $2\triangle^2$.

(b) The Poincaré dual of (a).

(c) An arrangement of tropical hyperplanes.

Figure 1: The correspondence between mixed subdivisions and tropical pseudohyperplane arrangements.

between classical problems and their tropical analogues, see *e.g.* [AB07; AK06; DS04; Mik06].

Moreover, tropical geometry has far reaching connections to other objects of discrete geometry: By Develin and Sturmfels [DS04] *regular* subdivisions of $\triangle^{n-1} \times \triangle^{d-1}$ are dual to arrangements of n tropical hyperplanes in $\mathbb{T}^{d-1}$. See Figure 1 for an illustration. By the *Cayley Trick* (*cf.* Huber, Rambau and Santos [HRS00]) subdivisions of $\triangle^{n-1} \times \triangle^{d-1}$ are in bijection with mixed subdivisions of $n\triangle^{d-1}$.

Here we will take a more combinatorial point of view and study the relationship of arrangements of tropical hyperplanes and mixed subdivisions of $n\triangle^{d-1}$: Combinatorially, a tropical hyperplane in $\mathbb{T}^{d-1}$ is just the (codimension-1-skeleton of the) polar fan of the $(d-1)$-dimensional simplex $\triangle^{d-1}$. For a $(d-2)$-dimensional tropical hyperplane H the d connected components of $\mathbb{TP}^{d-1} \setminus H$ are called the *(open) sectors* of H – they form the analogues to the sides $+/-$ of the classical oriented hyperplanes.

As in the classical situation, an arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$ induces a cell decomposition of $\mathbb{T}^{d-1}$ and each cell can be assigned a *type* or *tropical covector* that describes its position relative to each of the tropical hyperplanes. To be precise, the point p is assigned the type $A = (A_1, \dots, A_n)$ where $A_i \subseteq [d]$ denotes the set of closed sectors of the i-th tropical hyperplane in which p is contained. See Figure 1(c) for an illustration in dimension 2.

By [AD09, Theorem 6.3], the types of a tropical oriented matroid with parameters (n, d) yield a subdivision of $\triangle^{n-1} \times \triangle^{d-1}$. They also conjecture this to be a bijection, *i.e.,* they conjecture that the types of the cells in any mixed subdivision of $n\triangle^{d-1}$ are the types of a tropical oriented matroid with parameters (n, d).

In Oh and Yoo [OY11] the conjecture is proven for *fine* mixed subdivisions; in [H12b] we generalise this result to all mixed subdivisions of $n\triangle^{d-1}$.

In this paper we introduce arrangements of tropical pseudohyperplanes (see Definition 4.3) and prove one tropical analogue to the Topological Representation Theorem for (classical) oriented matroids [FL78].

A *tropical pseudohyperplane* is basically a set which is PL-homeomorphic to a tropical hyperplane (see also Definition 4.1). The challenging part is the definition of arrangements of such: We have to impose restrictions on the intersections of the pseudohyperplanes in the arrangement. In the classical framework, the intersections of the hyperplanes in the arrangement have to be homeomorphic to linear hyperplanes (of smaller dimension). In the tropical world, however, this approach is not feasible, since intersections of tropical hyperplanes are no longer homeomorphic to tropical hyperplanes (but have a very complicated geometry). We choose a different approach instead. We employ the fact that the covectors of tropical oriented matroids define mixed subdivisions of $n\triangle^{d-1}$ and impose restrictions on the cell decomposition induced by the tropical pseudohyperplanes in the arrangement. We this definition we prove the main result of this paper:

Theorem 1.1 (Topological Representation Theorem, *cf.* Theorem 4.4)**.** *Every tropical oriented matroid can be realised by an arrangement of tropical pseudohyperplanes.*

The paper is organised as follows: In Section 2 we briefly review the definition of tropical oriented matroids as given in [AD09]. Section 3 is dedicated to mixed subdivisions of dilated simplices. In particular, we prove some results strengthening the close relationship to tropical oriented matroids that are, to the best of our knowledge, not yet in the literature. Moreover, we show that a mixed subdivision of $n\triangle^{d-1}$ is uniquely determined by (the types of) its 0-cells (Proposition 3.8). In Section 4 we introduce tropical pseudohyperplane arrangements and prove a first version of the Topological Representation Theorem. Another version and a second (equivalent) definition of tropical pseudohyperplane arrangements are given in [H12b].

A joint extended abstract [H12a] of this and [H12b] has been presented at FPSAC 2012. Moreover, the results in this paper are also contained in [H12c].

2 Tropical Oriented Matroids

The following definitions are analogous to those in [AD09].

Definition 2.1. For $n, d \geq 1$ an (n, d)-*type* is an n-tuple $(A_1, \dots, A_n)$ of non-empty subsets of $[d]$.

An (n, d)-type A can be represented as a subgraph K_A of the complete bipartite graph $K_{n,d}$: Denote the vertices of $K_{n,d}$ by $N_1, \dots, N_n, D_1, \dots, D_d$. Then the edges of K_A are $\{\{N_i, D_j\} \mid j \in A_i\}$.

For convenience we will write sets like $\{1, 2, 3\}$ as 123 throughout this article.

In particular, the types of points relative to an arrangement of n tropical hyperplanes of dimension $d-1$ as described above are (n, d)-types.

A *refinement* of an (n, d)-type A with respect to an ordered partition $P = (P_1, \dots, P_k)$ of $[d]$ is the (n, d)-type $B = A|_P$ where $B_i = A_i \cap P_{m(i)}$ and $m(i)$ is the smallest index

where $A_i \cap P_{m(i)}$ is non-empty for each $i \in [n]$. A refinement is *total* if all B_i are singletons.

Given (n, d)-types A and B, the *comparability graph* $\mathcal{C}G_{A,B}$ is a multigraph with node set $[d]$. For $1 \leq i \leq n$ there is an edge for every $j \in A_i, k \in B_i$. This edge is undirected if $j, k \in A_i \cap B_i$ and directed $j \to k$ otherwise. (We consider the comparability graph as a graph without loops.) Note that there may be several edges (with different directions) between two nodes.

A *directed path* in the comparability graph is a sequence $e_1, e_2, \ldots, e_k$ of incident edges at least one of which is directed and all directed edges of which are directed in the "right" (*i.e.,* the same) direction. A *directed cycle* is a directed path whose starting and ending point agree. The graph is *acyclic* if it contains no directed cycle.

Definition 2.2. A *tropical oriented matroid* M (with parameters (n, d)) is a collection of (n, d)-types which satisfies the following four axioms:

- *Boundary*: For each $j \in [d]$, the type $(j, j, \ldots, j)$ is in M.
- *Comparability*: The comparability graph $\mathcal{C}G_{A,B}$ of any two types $A, B \in M$ is acyclic.
- *Elimination*: If we fix two types $A, B \in M$ and a position $j \in [n]$, then there exists a type C in M with $C_j = A_j \cup B_j$ and $C_k \in \{A_k, B_k, A_k \cup B_k\}$ for $k \in [n]$.
- *Surrounding*: If A is a type in M, then any refinement of A is also in M.

We call $d =: \operatorname{rank} M$ the *rank* and n the *size* of M.

Example 2.3. By [AD09, Theorem 3.6] the set of types of an arrangement of n tropical hyperplanes in $\mathbb{T}^{d-1}$ is a tropical oriented matroid with parameters (n, d).

We call tropical oriented matroids coming from an arrangement of tropical hyperplanes *realisable.* Recall that by Develin and Sturmfels [DS04] realisable tropical oriented matroids are in bijection with *regular* mixed subdivisions of $n\triangle^{d-1}$.

The axiom system was built to capture the features of the set of types in tropical hyperplane arrangements and thus the axioms have geometric interpretations:

The *boundary axiom* ensures that all tropical hyperplanes in the arrangement are embedded correctly into $\mathbb{TP}^{d-1} \cong \triangle^{d-1}$. The *surrounding axiom* describes what the neighbourhood of a point of type A (or equivalently, the star of the cell A in the cell complex) looks like. The *elimination axiom* describes the intersection of a tropical line segment from A to B with the j-th tropical hyperplane. Finally, the *comparability axiom* ensures that we can declare a "direction from A to B". Each position puts certain constraints on the direction vector, which may not contradict one another.

Definition 2.4. The *dimension* of an (n, d)-type A is the number of connected components of K_A minus 1. A *vertex* is a type of dimension 0, an *edge* a type of dimension 1 and a *tope* a type of full dimension $d - 1$, *i.e.,* each tope is an n-tuple of singletons.

For two types A, B we write $A \supseteq B$ if $A_i \supseteq B_i$ for each $i \in [n]$. Moreover, we define the intersection $A \cap B := (A_1 \cap B_1, \ldots, A_n \cap B_n)$ and union $A \cup B := (A_1 \cup B_1, \ldots, A_n \cup B_n)$.

A type A in a tropical oriented matroid M is *bounded* if all elements of $[d]$ appear in A and *unbounded* otherwise.

A realisable tropical oriented matroid is in general position if and only if the corresponding arrangement of tropical hyperplanes is so. Moreover, for a realisable tropical oriented matroid, the bounded types correspond to the bounded cells in the cell decomposition of $\mathbb{T}^{d-1}$.

Definition 2.5 (*Cf.* [AD09, Propositions 4.7 and 4.8])**.** Let M be a tropical oriented matroid with parameters (n, d).

1. For $i \in [n]$ the *deletion* $M_{\setminus i}$ consisting of all $(n-1, d)$-types which arise from types of M by deleting coordinate i is a tropical oriented matroid with parameters $(n-1, d)$.

2. For $j \in [d]$ the *contraction* $M_{/j}$ consisting of all types of M that do not contain j in any coordinate is a tropical oriented matroid with parameters $(n, d-1)$.

3 Mixed Subdivisions

Given two sets X, Y their *Minkowski sum* $X + Y$ is given by $X + Y := \{x + y \mid x \in X, y \in Y\}$.

Definition 3.1. Let $P_1, \ldots, P_k \subset \mathbb{R}^n$ be (full-dimensional) convex polytopes. Then a polytopal subdivision $\{Q_1, \ldots, Q_s\}$ of $P := \sum P_i$ is a *mixed subdivision* if it satisfies the following conditions:

1. Each Q_i is a Minkowski sum $Q_i = \sum_{j=1}^{k} F_{i,j}$, where $F_{i,j}$ is a face of P_j.

2. For $i, j \in [s]$ we have that $Q_i \cap Q_j = (F_{i,1} \cap F_{j,1}) + \ldots + (F_{i,k} \cap F_{j,k})$.

Note that this definition can easily be generalised for polytopes which are not full-dimensional.

Let S, S' be mixed subdivisions of $n\triangle^{d-1}$. Then we say that S' is a *refinement* of S if for every cell $C' \in S'$ there is a cell $C \in S$ such that $C' \subseteq C$. This defines a partial order on the set of mixed subdivisions of $n\triangle^{d-1}$. A mixed subdivision is *fine* if there is no mixed subdivision refining it. By Santos [San05, Proposition 2.3] this is equivalent to the condition that for every cell $B = \sum B_i$ all the B_i lie in mutually independent affine subspaces (and this is satisfied if and only if $\dim B = \sum \dim B_i$).

3.1 Mixed Subdivisions of $n\triangle^{d-1}$

We are interested in the case of mixed subdivisions where $P_i = \triangle^{d-1}$ for each i. Then $\sum P_i = n\triangle^{d-1}$ is a dilated simplex. By Ardila and Develin [AD09, Theorem 6.3] the types of a tropical oriented matroid with parameters (n, d) yield a mixed subdivision of $n\triangle^{d-1}$. A tropical oriented matroid is in general position if and only if its mixed subdivision is fine.

If $Q = \sum_{i=1}^{k} F_i$, where $F_i \subset [d]$ is a cell in such a mixed subdivision then we call $(F_1, F_2, \ldots, F_k)$ its *type* and denote it by $\mathcal{T}_Q$. Note that this is an (n, d)-type as defined in Definition 2.1. Conversely, if we are given an (n, d)-type A then this corresponds to a unique cell inside $n\triangle^{d-1}$, which we denote by $\mathcal{C}_A$.

In general, we call a cell corresponding to an (n, d)-type, *i.e.,* a Minkowski sum of n faces of $\triangle^{d-1}$, a *Minkowski cell.*

To avoid confusion with the vertices of tropical oriented matroids, we speak of the 0-dimensional cells of a mixed subdivision as *topes*.

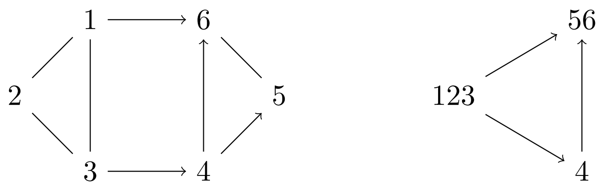
We now establish some properties of mixed subdivisions of $n\triangle^{d-1}$ – or more generally about (n, d)-types. Note that since we can describe the Minkowski cells in a mixed subdivision of $n\triangle^{d-1}$ in terms of (n, d)-types, we can transfer properties of tropical oriented matroids (such as the boundary, surrounding, comparability or elimination property) as defined in Section 2 to mixed subdivisions of $n\triangle^{d-1}$.

Lemma 3.2. *Let A, B be two (n, d)-types with $A \subseteq B$. Then A is a refinement of B if and only if $\mathcal{G}_{A,B}$ is acyclic.*

Note that we do not assume that the types in this lemma are contained in a tropical oriented matroid. In particular, there is a tropical oriented matroid containing both A and B if and only if $\mathcal{G}_{A,B}$ is acyclic.

Proof. First assume that $\mathcal{G}_{A,B}$ is acyclic. Let G be the directed graph obtained from $\mathcal{G}_{A,B}$ by contracting all undirected edges. This is well-defined and acyclic since $\mathcal{G}_{A,B}$ is acyclic. We will label the vertices of G by the according subsets of $[d]$. Let $P = (P_1, \ldots, P_\ell)$ be a linear extension of the partial order on the vertices of G that is defined by the edges. This process is illustrated in Figure 2. We will now argue that $B|_P = A$. Indeed by the definition of refinements, $(B|_P)_i$ contains all elements of B_i which come first in P. Since $A_i \subseteq B_i$, in $\mathcal{G}_{A,B}$ every element of A_i has an outgoing edge to each element of $B_i - A_i$. Hence in P the elements of A_i come before the elements of $B_i - A_i$. Moreover, the elements of A_i form a clique in $\mathcal{G}_{A,B}$ and are thus contained in the same P_i. This shows that $(B|_P)_i = A_i$ for each $i \in [n]$.

Conversely, assume that $A = B|_P$ for some ordered partition P of $[d]$. Consider the graph $H = ([d], E)$ with an undirected edge $\{i, j\}$ for each $i, j \in P_a$ and a directed edge $i \to j$ whenever $i \in P_a, j \in P_b$ with $a < b$. Then clearly H is acyclic. We now show that $\mathcal{G}_{A,B}$ is a subgraph of H, which completes the claim. Indeed let $i, j \in [d]$. If $\mathcal{G}_{A,B}$ has an undirected edge $\{i, j\}$ then there is $k \in [n]$ such that $i, j \in A_k \cap B_k$ and hence there is P_ℓ such that $i, j \in P_\ell$. On the other hand, if $\mathcal{G}_{A,B}$ has a directed edge $i \to j$ then



$P = 123, 4, 56$

Figure 2: Assume that in the proof of Lemma 3.2 we have $A = (123, 1, 3, 4, 56), B = (123, 16, 34, 456, 56)$. Then $A \subseteq B$ and $\mathcal{G}_{A,B}$ is the graph on the left. Then by contracting all undirected edges we obtain the graph G drawn in the center. By fixing a linear extension of this (in fact, there is only one in this example) we get the ordered partition of $[6]$ on the right hand side. Moreover, one easily verifies that indeed $A = B|_P$.

there is $k \in [n]$ such that $i \in A_k \cap B_k$ but $j \in B_k - A_k$. If we choose a, b such that $i \in P_a$ and $j \in P_b$ then we must have $a < b$. □

Lemma 3.3. *Let A, B be two types in a mixed subdivision S of $n\triangle^{d-1}$. Then their intersection $A \cap B$ either has an empty position or is also a type in S.*

Proof. Let A, B be two types that intersect non-trivially in every position. It is an easy exercise to verify that $\mathcal{G}_{A,A\cap B}$ is a subgraph of $\mathcal{G}_{A,B}$. Hence $\mathcal{G}_{A,A\cap B}$ is acyclic since $\mathcal{G}_{A,B}$ is so. By Lemma 3.2 this implies that $A \cap B$ is a type. □

Lemma 3.4. *Given a Minkowski cell $Q = \sum_{i=1}^{k} F_i$ in a mixed subdivision of $n\triangle^{d-1}$ then the faces of Q are exactly the $\mathcal{C}_R$ where R is a refinement of $\mathcal{T}_Q$.*

Proof. This follows directly from [AD09, Proposition 6.4]. □

Lemma 3.5. *Let A, B be (n, d)-types such that $\mathcal{G}_{A,B}$ is acyclic. Then $\mathcal{C}_A \cap \mathcal{C}_B = \mathcal{C}_{A\cap B}$.*

Proof. It is easy to see that the intersection of the cells $\mathcal{C}_A$ and $\mathcal{C}_B$ is always the convex hull of integral points (in the standard embedding into $\mathbb{R}^d$) in $n\triangle^{d-1}$. Moreover, it is clear that $\mathcal{C}_{A\cap B} \subseteq \mathcal{C}_A \cap \mathcal{C}_B$.

Conversely, let p be an integral point in $\mathcal{C}_A \cap \mathcal{C}_B$. Denote by $p_A \subseteq A$ a possible type of p (which need not be a refinement of A), *i.e.*, p_A is an (n, d)-type with $p = \mathcal{C}_{p_A}$.We will now argue that then also $p_A \subseteq B$. So suppose this is not true. Define p_B similarly to p_A. Then p_B is a permutation of p_A. Hence $\mathcal{G}_{p_A,p_B}$ contains a directed cycle C.

But then C is also contained in $\mathcal{G}_{A,B}$ (where some directed edges in $\mathcal{G}_{p_A,p_B}$ may be undirected in $\mathcal{G}_{A,B}$). But since $p_A \not\subseteq B$ there is at least one directed edge. This contradicts the hypothesis that $\mathcal{G}_{A,B}$ is acyclic. □

We can define the concepts of *deletion* and *contraction* for mixed subdivisions analogous to Definition 2.5. The following observations are immediate:

Lemma 3.6. *Let S be a mixed subdivision of $n\triangle^{d-1}$.*

1. *For any $i \in [n]$ the deletion $S_{\setminus i}$ is a mixed subdivision of $(n-1)\triangle^{d-1}$.*

2. *For any $j \in [d]$ the contraction $S_{/j}$ is a mixed subdivision of $n\triangle^{d-2}$.*

Proof.

1. This follows immediately from Santos [San05, Lemma 2.1].

2. The contraction $S_{/j}$ is the subdivision of the j-th facet of $n\triangle^{d-1}$ (*i.e.,* the facet opposite to the vertex $(j, \ldots, j)$) induced by S. Hence $S_{/j}$ is a mixed subdivision.

□

There is a standard embedding of a mixed subdivision of $n\triangle^{d-1}$ into $\mathbb{R}^d$ (by mapping a tope v to $(x_1, \ldots, x_d)$ where x_i is the number of occurences of i in v). We thus regard a mixed subdivision – or any subset of its (open) cells – as a metric space with the Euclidean metric inherited from $\mathbb{R}^d$. The following is immediate:

Lemma 3.7. *Let S be a mixed subdivision of $n\triangle^{d-1}$, $i \in [n], j \in [d]$. Let X be the subcomplex of S of all cells A such that $A_i = j$. Then X is embedded isometrically into the deletion $S_{\setminus i}$.*

3.2 Reconstructing Mixed Subdivisions

In this section we prove the following:

Proposition 3.8. *Let S be a mixed subdivision of $n\triangle^{d-1}$. Then S can be reconstructed from its topes.*

More precisely, the cells of S are exactly the unions of topes all of whose total refinements are topes and which do not contain any other tope.

We call types satisfying the conditions above the *nice* types of S. *I.e.,* an (n, d)-type A is nice if

- A is a (componentwise) union of topes of S,
- all total refinements of A are topes of S, and
- if T is a tope of S such that $T \subseteq A$ then T is a refinement of A.

If A is a nice type we call the Minkowski cell $\mathcal{C}_A$ corresponding to A a *nice cell.*

Note that it is crucial to consider the topes of S as types rather than as mere coordinates; *i.e.,* the order of the summands does matter.

Also note that the equivalent result for tropical oriented matroids, namely that a tropical oriented matroid is uniquely determined by its topes, is proven in [AD09]. Their proof, however, uses the elimination property.

Proof. Let S be a mixed subdivision of $n\triangle^{d-1}$. It is clear that all cells of S are nice. So it remains to prove that every nice type does indeed yield a cell of S.

The general strategy is the following: Assume that a cell A corresponds to a nice type of S. We proceed via induction over $\dim A$. If $\dim A = 0$ then it is clear that A is a cell of S (namely a vertex). Thus, we may assume that $\dim A \geq 1$ and that every proper refinement of A is a cell.

We will argue that A intersects every cell B of S either not at all or in a common face of A and B, proving that A is in fact a cell in S.

We may without loss of generality assume that A contains all elements of $[d]$. Otherwise form contractions of S for each element of $[d]$ that is not contained in A. Moreover, we may assume that A does not contain any singleton position. Otherwise form the deletion of S for every singleton position. By Lemma 3.7 A embeds isometrically into this deletion.

Now let B be a cell in S. By Lemma 3.5 it suffices to prove that A and B are comparable. So suppose on the contrary that $\mathcal{G}_{B,A}$ has a directed cycle.

Assume without loss of generality that this cycle is $C = (1, 2, \ldots, k, 1)$, directed in this order. Let $P = ([k], k+1, \ldots, d)$ be an ordered partition of $[d]$. Define $A' := A|_P$. Since A does not have any singleton positions, $\dim A' < \dim A$ if $k < d$ and hence A' is a proper refinement of A. Moreover, $\mathcal{G}_{B,A'}$ also contains the cycle C. This is a contradiction.

Thus, $k = d$. Assume without loss of generality that $B_i \ni i$ and $A_i \ni (i+1) \bmod d$ for each i. Since A does not have any singleton positions this implies that $A_i = \{i, i+1 \bmod d\}$ for each i. Moreover, $B_i = \{i\}$ if there is a directed edge $i \to (i+1 \bmod d)$ and $B_i = \{i, i+1 \bmod d\}$ if the edge is undirected. Thus, we have completely determined A and B.

Since the cycle is directed, there is a singleton in B. Assume without loss of generality that $B_d = \{d\}$. Let $P = (1, 2, \ldots, d)$ be an ordered partition of $[d]$ into singletons. Then $T := B|_P = (1, 2, \ldots, d)$. Hence T is a tope in S. But T is contained in A and not a refinement of A. This contradicts the choice of A. See Figure 3 for an illustration. $\square$

Since in a *fine* mixed subdivision the type graph of every type is acyclic, we get the following:

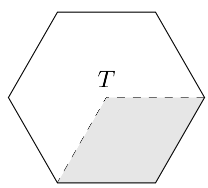


Figure 3: A (hexagonal) Minkowski cell A of type $(12, 23, 13)$ and a cell B of type $(12, 2, 13)$ as in the proof of Proposition 3.8. Then $A \supset B$ and there is a tope $T = (1, 2, 3)$ of B that lies in the interior of A.

Corollary 3.9. *Let S be a fine mixed subdivision of $n\triangle^{d-1}$. Then the (type graphs of the) cells of S are exactly the acyclic unions of (the type graphs of) topes all of whose total refinements are again topes.*

For $i \in [n]$ consider the *deletion map*

$$\cdot_{\setminus i} : S \to S_{\setminus i} : C \mapsto C_{\setminus i} = (C_1, \ldots, \widehat{C_i}, \ldots, C_n)$$

mapping each cell C of S to the cell obtained by omitting the i-th entry of C.

Lemma 3.10. *Let S be a mixed subdivision of $n\triangle^{d-1}$, $i \in [n]$ and $A \neq B$ the types of cells $\mathcal{C}_A, \mathcal{C}_B \in S$ cells such that $A_{\setminus i} = B_{\setminus i}$. Then $A \cup B$ is the type of a cell in S.*

Proof. Let $C := A \cup B$, *i.e.*, $C_i := A_i \cup B_i$ and $C_j = A_j (= B_j)$ for $j \neq i$. The situation is sketched in Figure 4. The intuition is that C (unless it already equals A or B) is a prism over A (or B) with A and B the top, respectively bottom face of C.

We need to show that C is indeed a cell in S. To this end, we verify that C satisfies the conditions from Proposition 3.8. This means we have to show that the total refinements of C are exactly the total refinements of A and B.

Indeed let $v = C|_P$ be a total refinement of C and assume without loss of generality that $v_i \in A_i$. Then $v = A|_P$ is also a total refinement of A. Conversely, let $v = A|_P$ be a total refinement of A. We may assume that in P the element v_i comes before all elements of $B_i \setminus A_i$. Otherwise we may change this order since $\mathcal{G}_{A,B}$ is acyclic. But then $C|_P = v$. Thus, every total refinement of A or B is also one of C. Hence C is a type in S. □

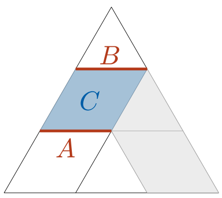


Figure 4: The two edges A and B are mapped to the same cell under the deletion map that deletes the shaded cells.

4 The "Tropological Representation Theorem"

In this section we formally introduce tropical pseudohyperplanes and prove a first version of the Topological Representation Theorem.

Definition 4.1. A *tropical pseudohyperplane* is the image of a tropical hyperplane under a PL-homeomorphism of $\mathbb{TP}^{d-1}$ that fixes the boundary.

The following theorem is a crucial ingredient to the proof of the Topological Representation Theorem. In an arrangement of tropical hyperplanes, the i-th tropical hyperplane consists exactly of those points A with $\#A_i \geq 2$. We show that the analogue holds for the Poincaré dual of a mixed subdivision of $n\triangle^{d-1}$. We denote the dual cell of a cell $C \in S$ by C^*. See again Figure 1(b) for an example.

Theorem 4.2. *Let S be a mixed subdivision of $n\triangle^{d-1}$ and $i \in [n]$. Then $\{C^* \mid C \in S, \#C_i \geq 2\}$ is a tropical pseudohyperplane.*

Proof. We prove the claim by induction over n. For $n = 1$ this is true since then $S = \triangle^{d-1}$ is the trivial subdivision, whose dual is the cell complex of one $(d-2)$-dimensional tropical hyperplane in $\mathbb{T}^{d-1}$.

Now assume $n \geq 2$. Choose $i \neq j \in [n]$ and consider the deletion $S_{\setminus j}$. By Lemma 3.6 this is a mixed subdivision of $(n-1)\triangle^{d-1}$ and by induction the image of H_i in $S_{\setminus j}$ is a tropical pseudohyperplane h.

But H_i is the preimage of h under the deletion map. By Lemma 3.10 this preimage is PL-homeomorphic to h and hence a tropical pseudohyperplane. $\square$

4.1 Arrangements of tropical pseudohyperplanes I

In this section we suggest one definition for tropical pseudohyperplane arrangements. Note that another (equivalent) definition is given in [H12b].

Definition 4.3. An *arrangement of tropical pseudohyperplanes* is a finite family of tropical pseudohyperplanes such that

- the tropical pseudohyperplanes induce a regular subdivision of $\mathbb{T}^{d-1}$,
- in the cell decomposition the points of equal type form a PL-ball (in particular, there are no two cells with the same type),
- the types satisfy the surrounding and comparability property and
- the bounded cells are exactly those which correspond to bounded types.

The following theorem is the main result of this paper and can be seen as a first version of the Topological Representation Theorem for tropical oriented matroids.

Theorem 4.4 (*Tropological Representation Theorem*, Version I)**.** *Let $n, d \geq 1$. The Poincaré dual of a mixed subdivision of $n\triangle^{d-1}$ is a tropical pseudohyperplane arrangement as defined in Definition 4.3. Conversely, the dual of the cell decomposition of an arrangement of n tropical pseudohyperplanes in $\mathbb{TP}^{d-1}$ is a mixed subdivision of $n\triangle^{d-1}$.*

Proof. Let S be a mixed subdivision of $n\triangle^{d-1}$. By Theorem 4.2 and [AD09, Proposition 6.4], it is clear that S satisfies the axioms in Definition 4.3 above.

Conversely, let $\mathcal{A}$ be an arrangement of tropical pseudohyperplanes in $\mathbb{T}^{d-1}$ as in Definition 4.3. We have to show that the types of the cells in the induced cell decomposition

yield a mixed subdivision of $n\triangle^{d-1}$. So let $S := \{\mathcal{C}_A \mid A \text{ type in the cell complex of } \mathcal{A}\}$. Then S is a set of Minkowski cells in $n\triangle^{d-1}$.

By Lemmas 3.4 and 3.5, S is a polytopal complex whose realisation is contained in $n\triangle^{d-1}$. It remains to show that S covers $n\triangle^{d-1}$. We will use the fact that the 1-skeleton of $\mathcal{A}$ is path-connected.

To this end, let $\mathcal{C}_A$ be a maximal cell in S and let $\mathcal{C}_B$ be a facet of $\mathcal{C}_A$. Then A corresponds to a vertex in $\mathcal{A}$ and B corresponds to an edge containing A. The cell $\mathcal{C}_B$ is contained in the boundary of $n\triangle^{d-1}$ if and only if B is unbounded. In this case B is an unbounded edge in $\mathcal{A}$. If $\mathcal{C}_B$ is not on the boundary then there is a unique other maximal cell $\mathcal{C}_{A'}$ "on the other side" of $\mathcal{C}_B$, the other endpoint of B. Thus, S covers the whole of $n\triangle^{d-1}$. □

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